

On the monotonicity of the geo-logarithmic price indexes

Monotonia degli indici dei prezzi geo-logaritmici

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Riassunto: Nel presente lavoro si affronta il problema della monotonia degli indici dei prezzi appartenenti alla famiglia geo-logaritmica. Si dimostra che l'assioma di monotonicità non è in generale rispettato, fornendo una condizione necessaria e sufficiente per la sua validità.

Keywords: price index numbers, monotonicity, axiomatic approach.

1. Introduction

The geo-logarithmic family of price indexes (the so called P_{xy} family) was introduced in the early '90s in the framework of the axiomatic index number theory. Its parametrization allows a parsimonious representation of many known price indexes and guarantees that its members are well-behaved from an axiomatic point of view (Martini, 1992, 2001). Nevertheless, not all the axioms usually referred to in the context of axiomatic index number theory are fulfilled by geo-logarithmic indexes. Here we discuss the validity of the monotonicity axiom, proving a simple necessary and sufficient condition for a geo-logarithmic price index to be monotonic. In the following, we introduce the definition of the geo-logarithmic P_{xy} family, state the monotonicity axiom and prove the condition of monotonicity. A few examples conclude the paper.

2. Definition of the P_{xy} family

Let p_a , p_b , q_a and q_b be four n -dimensional vectors whose (non negative) components represent respectively the prices and the quantities of n goods in two different situations a e b . We want to compare the price in the situations a and b , assuming b as the basis, i.e. as the reference situation.

For $0 \leq x, y \leq 1$, let q_x and q_y be two n -dimensional vectors whose components are the following means:

$$q_{xi} = q_{ai}^x q_{bi}^{1-x}; \quad q_{yi} = q_{ai}^y q_{bi}^{1-y} \quad i = 1, \dots, n \quad (1)$$

and let

$$w_{xi} = \frac{p_{ai} q_{xi}}{\sum_{i=1}^n p_{ai} q_{xi}}; \quad w_{yi} = \frac{p_{bi} q_{yi}}{\sum_{i=1}^n p_{bi} q_{yi}}. \quad (2)$$

The P_{xy} family is the family of the price indexes defined by:

$$P_{xy}(p_a, p_b, q_a, q_b) = \prod_{i=1}^n \left(\frac{p_{ai}}{p_{bi}} \right)^{\frac{\tau(w_{xi}, w_{yi})}{\sum_{j=1}^n \tau(w_{xj}, w_{yj})}} \quad 0 \leq x, y \leq 1 \quad (3)$$

where $\tau(s, t)$ is the logarithmic mean, defined as:

$$\tau(s, t) = \frac{s - t}{\ln s - \ln t} \quad s, t > 0, \quad s \neq t \quad (4)$$

with $\tau(s, s) = s$.

The P_{xy} family contains many “famous” indexes. For example, P_{10} is the Sato-Vartia price index, P_{00} is the Laspeyres price index, P_{11} is the Paasche price index and $P_{0.5\,0.5}$ is the Walsh price index. However, there are also well known price indexes not belonging to this family, the most important example being the celebrated Fisher index.

From (1) and (3), it follows that the P_{xy} indexes satisfy the following identity, that could be adopted as an alternative definition of the P_{xy} family as well:

$$P_{xy}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) = P_{10}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{q}_x, \mathbf{q}_y). \quad (5)$$

In other words, any P_{xy} index is nothing but the Sato-Vartia price index computed on the virtual quantities $\mathbf{q}_x$ and $\mathbf{q}_y$.

3. The P_{xy} family and the monotonicity axiom

Let $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ be two n -dimensional price vectors. We say the $\mathbf{p}^{(1)} > \mathbf{p}^{(2)}$ if $\mathbf{p}_i^{(1)} \geq \mathbf{p}_i^{(2)}$ for every $i = 1, \dots, n$, the inequality being strict for at least one i .

A price index is said to be *monotonic* if it satisfies the following *monotonicity axiom* (Balk, 1995):

Monotonicity axiom:

$$\mathbf{p}_a^{(1)} > \mathbf{p}_a^{(2)} \Rightarrow P_{xy}(\mathbf{p}_a^{(1)}, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) > P_{xy}(\mathbf{p}_a^{(2)}, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) \quad (6)$$

$$\mathbf{p}_b^{(1)} > \mathbf{p}_b^{(2)} \Rightarrow P_{xy}(\mathbf{p}_a, \mathbf{p}_b^{(1)}, \mathbf{q}_a, \mathbf{q}_b) < P_{xy}(\mathbf{p}_a, \mathbf{p}_b^{(2)}, \mathbf{q}_a, \mathbf{q}_b) \quad (7)$$

for any fixed price vectors $\mathbf{p}_a, \mathbf{p}_b$ and quantity vectors $\mathbf{q}_a$ and $\mathbf{q}_b$. In other words, a price index is monotonic if it increases as $\mathbf{p}_a$ increases and decreases as $\mathbf{p}_b$ increases.

When the geo-logarithmic family was introduced, it was argued to be composed of monotonic indexes. Actually this is not true, because the Sato-Vartia index is not monotonic, as shown in the work of Olt (Olt, 1996) and fully discussed in the paper of Dorfman and Reinsdorf (Dorfman and Reinsdorf, 1999). With the following proposition, we state a necessary and sufficient condition for indexes in the P_{xy} family to be monotonic.

Proposition. An element of the P_{xy} family is monotonic if and only if $x = y$.

Proof. *Sufficiency.* Let us first show that the so called *diagonal* subfamily P_{xx} is composed of monotonic indexes only. An element of this subfamily has the following form:

$$P_{xx}(\mathbf{p}_a, \mathbf{p}_b, \mathbf{q}_a, \mathbf{q}_b) = \frac{\sum_{i=1}^n \mathbf{p}_{ai} \mathbf{q}_{xi}}{\sum_{i=1}^n \mathbf{p}_{bi} \mathbf{q}_{xi}} = \frac{\sum_{i=1}^n \mathbf{p}_{ai} \mathbf{q}_{ai}^x \mathbf{q}_{bi}^{1-x}}{\sum_{i=1}^n \mathbf{p}_{bi} \mathbf{q}_{ai}^x \mathbf{q}_{bi}^{1-x}} \quad (8)$$

as can be easily verified from (3) with a direct computation. Clearly:

$$\frac{\partial P_{xx}}{\partial p_{aj}} \geq 0 \quad \forall j = 1, \dots, n \quad (9)$$

and

$$\frac{\partial P_{xx}}{\partial p_{bj}} \leq 0 \quad \forall j = 1, \dots, n. \quad (10)$$

Thus P_{xx} is monotonic for every $x \in [0, 1]$.

Necessity. Now, let $x \neq y$. We show that in this case no P_{xy} index is monotonic, showing that the non-monotonicity of the Sato-Vartia index implies the non-monotonicity of any index P_{xy} . For sake of simplicity, we discuss the case of two goods only (we could pursue the discussion in full generality, but this simpler case is enough for our purpose).

Let z_a and z_b two quantity vectors such that the Sato-Vartia index violates the monotonicity axiom. If q_a and q_b are such that $q_x = z_a$ and $q_y = z_b$, then the index P_{xy} is not monotonic too, as from (5):

$$P_{xy}(p_a, p_b, q_a, q_b) = P_{10}(p_a, p_b, z_a, z_b). \quad (11)$$

So we just need to show that the following equation systems

$$\begin{cases} q_{ai}^x q_{bi}^{1-x} = z_{ai} \\ q_{ai}^y q_{bi}^{1-y} = z_{bi} \end{cases} \quad i = 1, 2 \quad (12)$$

have a solution whenever $x \neq y$. To this goal, it is enough to observe that:

$$q_{ai} = z_{ai}^{\frac{y}{y-x}} z_{bi}^{\frac{x}{x-y}}, \quad q_{bi} = z_{ai}^{\frac{y-1}{y-x}} z_{bi}^{\frac{x-1}{x-y}} \quad i = 1, 2 \quad (13)$$

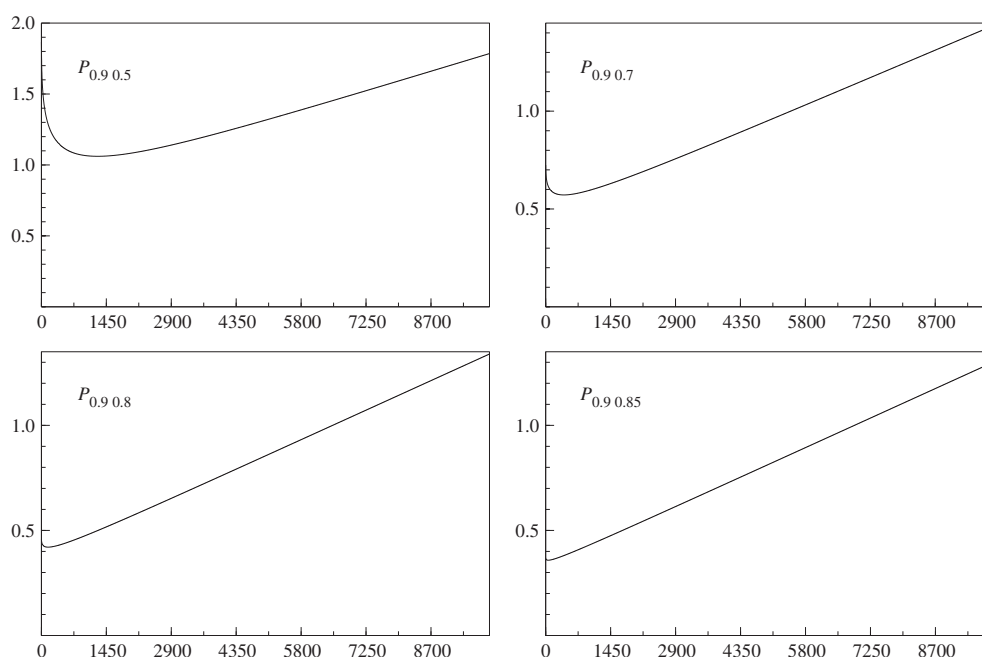
are the explicit solutions of the systems (note that the condition $x \neq y$ is required for the solutions to exist).

□

4. A few examples

In Figure 1 we show the non-monotonicity with respect to p_{a1} of the P_{xy} indexes, for $x = 0.9$ and $y = 0.5, 0.7, 0.8$, and 0.85 respectively, in the simple case where two goods are considered. Here, the horizontal axis represents p_{a1} and the vertical axis represents the value of the price index. Note that the values of p_{a1} corresponding to the minimum approach zero when x approaches y (coherently with the monotonicity of the subfamily P_{xx}). For the simulation, we choose $p_{a2} = 100$, $p_{b1} = p_{b2} = 10$, $q_{a1} = 300$, $q_{a2} = 5$, $q_{b1} = 1$ and $q_{b2} = 8$.

Figure 1: Non-monotonicity of some geo-logarithmic price indexes.



5. Conclusion

We have proved that geo-logarithmic indexes do not meet the monotonicity axiom, unless they belong to the subfamily P_{xx} . This result does not imply that geo-logarithmic indexes outside the diagonal subfamily should be rejected. Rejection of such indexes depends on the relevance that we give to the monotonicity axiom in the specific comparison problem we are dealing with. For example, in the context of cost of living measures, not all the authors agree on the necessity of the monotonicity axiom (see Dorfman and Reinsdorf, 1999). Anyway, our interest was theoretical in nature and our simple result contributes to clarify the properties of the geo-logarithmic family of price indexes.

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